Finite Permutation Groups

1. The Basic Notions

Doing mathematics from the simplest things.

a finite set X, an one-to-one map M from X to X means a permutation on X.

({all of such maps}, composition of maps) is a non-Abell group, named symmetric group on X, Sym(X).

complete induction

any $M : X \to X$ can be expressed as a composition of some special maps: rotations, such a rotation can be express as:

$$(x_1, x_2, ..., x_k), means : M(x_1) = x_2, M(x_2) = x_3, ..., M(x_k) = x_1.$$

and any two rotations won't treat of the same element of X. so, such composition of rotations is commutative, then, the form that a permutation is expressed by a composition of some rotations is unique.

any rotation can be expressed as a composition of some transpositions :(x_1, x_2). means

$$(x_1, x_2, ..., x_k) = (x_1, x_k)(x_1, x_{k-1})...(x_1, x_2).$$

such a composition is not commutative.

then, any M can be expressed as a composition of some transpositions, and because of the composition of rotations is commutative, the form that a permutation is expressed by a composition of some transpositions is NOT unique.

but, if

$$M = \tau_1\tau_2...\tau_k = \sigma_1\sigma_2...\sigma_l,$$

we have

$$k \equiv l(mod2).$$

then, if a permutation can be expressed as a composition of a even number of transpositions, name it an even permutation, otherwise, an odd permutation.

if we denote an even permutation M with sgn(M)=1, and an odd permutation with sgn(M)=-1, then sgn is an homomorphism from Sym(X) to group ({-1,1},*), the kernel of the homomorphism is the set of all the even permutations, and is an normal subgroup of Sym(X), name it a alternating group, denote it with Alt(X).

so, we naturally get the first subgroup of Sym(X).

now,we consider any general subgroup G of Sym(X). as a group, it act on X too, so we denote such relation as (G,X).